

Superdense coding using the quantum superposition principle

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By sending a classical two-level system, one can transfer information about only *two* distinguishable outcomes. Here we show that in quantum mechanics, using both the spin and path degrees of freedom of a spin-1/2 particle, and a Mach-Zehnder type interferometric arrangement with two suitable Stern-Gerlach detectors, it is possible to transfer information about *four* distinguishable outcomes. This procedure does *not* require using quantum entanglement as a resource as in the well-known protocol of dense coding, but instead hinges entirely on the quantum superposition principle. We also study probabilistic dense coding using our set-up and show that the dense coding scheme using quantum superposition cannot be optimized any further by extending the interferometric arrangement with more beam splitters.

Introduction

The quantum mechanical description of the nature of physical systems is enigmatic, and continues to spring surprises. On the one hand, it baffles specialists with its interpretational puzzles to do with measurement [1], nonlocality [2] and information [3]. On the other hand, it fascinates all with the unfolding of novel features associated with the superposition principle, as well as the applications of quantum entanglement. Quantum superposition is responsible for several fascinating phenomena such as Bose-Einstein condensation [4] and superfluidity [5]. Entanglement is the key ingredient for information processing protocols such as error correction [6] and teleportation [7].

Much is advocated about the utility of quantum entanglement as resource for performing tasks impossible through classical means, and quite justly so. But it is rather important to demarcate the useful tasks that may indeed be possible with the aid of the linear structure of quantum mechanics embossed by the superposition principle without taking recourse to the entanglement of two or more particles. In this context it is worthwhile to recall the debate on the necessity of using entangled pairs of particles for key generation in quantum cryptography [8, 9]. Such distinction of the applications of purely quantum superposition, apart from its interesting pedagogical aspects, could be of operational significance too, since it might be considerably harder to set up and maintain entanglement at the practical level.

The transfer of information in quantum theory through superdense coding [10] is regarded as one of the major applications of quantum entanglement. In contrast, in this work we demonstrate the possibility of superdense coding employing the superposition principle for a single particle using its path and spin degrees of freedom.

This indeed is an example of a surprising application of quantum mechanics, since it is contrary to the widely subscribed notion of the entanglement of two particles being essential for dense coding [10, 11].

Classically, by sending a two-level system one can transfer information about only two distinguishable outcomes. In other words, only one bit may be encoded in one spin-1/2 particle. However, it is well-known that by the use of entanglement, the technique of dense coding [10] is able to transfer information about four distinguishable outcomes by sending a two-level system that is prior entangled with another two-level system with the receiver. In this technique, Alice and Bob share the two members of an EPR pair of states. Alice then codes the required information into the spin-state of her member of the pair, and then sends this particle to Bob who carries out a Bell-basis measurement to obtain the information. The technique demonstrates that shared entanglement can enable Alice and Bob to enhance the capacity of a shared quantum channel up to the Holevo limit [12] by transmitting two bits of information using two qubits. Mattle et al [13] have put the technique into practice using polarisation-entangled photons, and other important work on dense coding includes a fundamental discussion of Mermin [11], a dense coding protocol for continuous variables due to Braunstein and Kimble [14], and achievement of this scheme by Li et al [15]. Full experimental distinction between all four Bell states is still an ongoing task [16].

It is therefore interesting to explore the viability of an alternative scheme for transferring information about four distinguishable outcomes which does not require quantum entanglement as a resource, but rather relies on the quantum superposition principle. The scheme presented in this paper utilises both the spin and path

degrees of freedom of a spin-1/2 particle. The discussion of the scheme is in terms of spin-1/2 particles, such as neutrons, and uses a variant of the Mach-Zehnder interferometer, with suitable manipulations of both the spin and path degrees of freedom. However the scheme works equally well for photons with appropriate polarising and analysing devices.

The setup

In the variant of the Mach-Zehnder interferometer we are using here (Fig.1), an input spin-1/2 particle with an initial spin polarised state $|\uparrow\rangle_z$ is first passed through a spin rotator (SR) (in which a uniform magnetic field is directed along the \hat{x} -axis) before it is incident on the first beam splitter (BS1) of this setup (Fig.1). The action of SR is to change the initial spin state $|\uparrow\rangle_z = \frac{1}{\sqrt{2}}(|\rightarrow\rangle_x + |\leftarrow\rangle_x)$ to the state, say, $|\chi\rangle$ given by

$$|\chi\rangle = \frac{1}{\sqrt{2}}(|\rightarrow\rangle_x + e^{i\delta}|\leftarrow\rangle_x) \quad (1)$$

where δ is the relative phase shift between $|\rightarrow\rangle_x$ and $|\leftarrow\rangle_x$ introduced by SR. In passing through BS1 with both the reflection and transmission probabilities 1/2, the input particle can emerge along either the transmitted or the reflected channel. The state of the emergent particle in either of these channels corresponds respectively to either $|\psi_1\rangle$ or $|\psi_2\rangle$ which have a relative phase shift of $(\pi/2)$ between them arising because of the reflection from BS1. Note that, $|\psi_1\rangle$ and $|\psi_2\rangle$ are eigenstates of the projections operators $P(\psi_1)$ and $P(\psi_2)$ respectively, which pertain to measurements determining ‘which channel’ the particle is in. For example, the results of such measurements for the transmitted (reflected) channel with binary alternatives are given by the eigenvalues of $P(\psi_1)$ ($P(\psi_2)$); the eigenvalue +1 (0) corresponds to the particle being found (not found) in the channel represented by $|\psi_1\rangle$ ($|\psi_2\rangle$).

Next, a ‘path’ phase shifter (PS) is applied along one of the channels, say $|\psi_2\rangle$, that introduces a relative phase shift, say ϕ , between the states $|\psi_1\rangle$ and $|\psi_2\rangle$. Reflections from the two mirrors M1 and M2 do not lead to any net relative phase shift between the states $|\psi_1\rangle$ and $|\psi_2\rangle$.

After all the above mentioned operations, the total state is given by the spin state and the superposition of the two path states given by

$$|\Psi\rangle_{SR+PS} = \frac{1}{\sqrt{2}}(|\psi_1\rangle + ie^{i\phi}|\psi_2\rangle)|\chi\rangle \quad (2)$$

Subsequently, a second beam splitter (BS2) is used whose reflection and transmission probabilities are 1/2. After emerging from BS2, the total state is given by

$$|\Psi\rangle_{BS2} = \frac{1}{2}[i|\psi_3\rangle(1 + e^{i\phi}) + |\psi_4\rangle(1 - e^{i\phi})]|\chi\rangle \quad (3)$$

where $\langle\psi_3|\psi_4\rangle = 0$, while the output ‘path’ states $|\psi_3\rangle$ and $|\psi_4\rangle$ are unitarily related to the states $|\psi_1\rangle$ and $|\psi_2\rangle$

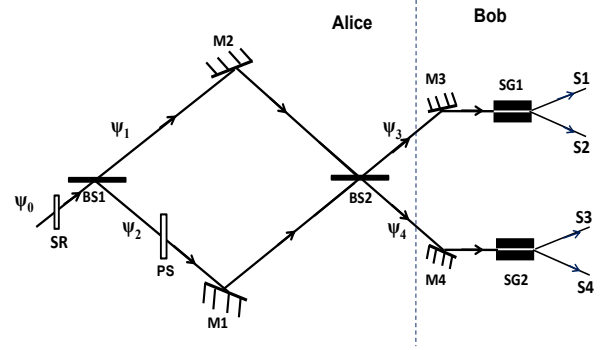


FIG. 1: A spin-1/2 particle (say, a neutron) with an initial spin polarised state $|\uparrow\rangle_z$ is first passed through a spin rotator (SR) before entering this Mach-Zehnder type setup through a beam splitter BS1. A phase shifter (PS) is placed along the channel $|\psi_2\rangle$. The relevant spin measurements are considered on the neutron emerging from the beam splitter BS2 by using the two spatially separated Stern-Gerlach devices SG1 and SG2. The two output channels of SG1 (placed along the channel $|\psi_3\rangle$) are denoted S1 and S2, and similarly, the two output channels of SG2 (placed along the channel $|\psi_4\rangle$) are denoted by S3 and S4.

by the following relations

$$|\psi_1\rangle \rightarrow \frac{1}{\sqrt{2}}[i|\psi_3\rangle + |\psi_4\rangle]; \quad |\psi_2\rangle \rightarrow \frac{1}{\sqrt{2}}[i|\psi_4\rangle + |\psi_3\rangle] \quad (4)$$

Finally, the relevant spin measurements are considered for the particle emerging from BS2 by using the Stern-Gerlach devices SG1 and SG2 placed along the channels $|\psi_3\rangle$ and $|\psi_4\rangle$ respectively, with the inhomogeneous magnetic field within these devices oriented along the $+\hat{z}$ -axis. This completes the description of the setup that is required for the information transfer scheme considered here.

The scheme

We now explain how this scheme works. For sending information to Bob, the state given by Eq.(2) is the state Alice prepares by suitably adjusting the parameters δ and ϕ corresponding to spin and path degrees of freedom respectively. In order to illustrate the present scheme, we consider two possible choices for each of the parameters δ and ϕ . Thus, there are four possible combinations of choices, each choice corresponding to a combined unitary operation performed by Alice. These four possible unitary operations are denoted by $U1$, $U2$, $U3$ and $U4$ where $U1$ corresponds to $\delta = 0, \phi = 0$, and similarly $U2$ ($\delta = 0, \phi = \pi$), $U3$ ($\delta = \pi, \phi = 0$) and $U4$ ($\delta = \pi, \phi = \pi$).

Now, suppose Alice wants to communicate a certain outcome by subjecting the particle with her to a specific

unitary operation, say, $U1$. Then, from Eq.(3), it follows that the particle communicated to Bob ends up being in the channel $|\psi_3\rangle$ with the spin state $|\uparrow\rangle_z$. Similarly, for the respective unitary operations $U2$, $U3$ and $U4$, in any given case, the particle ends up in one of the two channels $|\psi_3\rangle$ and $|\psi_4\rangle$, with one of the two spin states $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$. All these four possibilities are encapsulated as follows

$$|\Psi^{U1}\rangle_{BS2} = |\psi_3\rangle |\uparrow\rangle_z; \quad |\Psi^{U2}\rangle_{BS2} = |\psi_3\rangle |\downarrow\rangle_z \quad (5)$$

$$|\Psi^{U3}\rangle_{BS2} = |\psi_4\rangle |\uparrow\rangle_z; \quad |\Psi^{U4}\rangle_{BS2} = |\psi_4\rangle |\downarrow\rangle_z \quad (6)$$

Thus, in order to discern the communicated information, Bob has to perform spin measurements with SG1 and SG2. Let us denote the two output channels of SG1 (placed along the channel $|\psi_3\rangle$) by S1 and S2, and similarly, the two output channels of SG2 (placed along the channel $|\psi_4\rangle$) are denoted by S3 and S4.

It is then seen from Eqs.(5,6) that corresponding to any one of Alice's four combinations of unitary operations $U1$, $U2$, $U3$ or $U4$, the communicated particle is detected by Bob with certainty in one of the four channels S1, S2, S3 or S4. This feature therefore enables our proposed scheme to be used for sending information about *four* distinguishable outcomes via a single particle, essentially by using only the superposition principle *without* requiring the use of any entangled state. Note here, that though this scheme is accomplished with a single particle, the particle essentially carries two 'qubits' in the form of a two-level spin state and a similar "two-level" (distinguishable) superposed path state.

Now, the question might arise as to whether it could be possible to encode information about *more than four* distinguishable outcomes using any variant of this scheme, e.g., by creating more superpositions by using more beam splitters. We will show here that if one more beam splitter is used, it is *not possible* to increase the information capacity further.

To this end let a 'path' phase shifter be applied along one of the channels, say along $|\psi_3\rangle$. This operation introduces a phase η . The beams are then reflected by mirrors $M3$ and $M4$, respectively, and reach the third beam splitter BS3. The beam splitter BS3 transforms the path state vectors $|\psi_3\rangle$ and $|\psi_4\rangle$ as

$$|\psi_3\rangle \rightarrow \frac{1}{\sqrt{2}}(i|\psi_5\rangle + |\psi_6\rangle); \quad |\psi_4\rangle \rightarrow \frac{1}{\sqrt{2}}(|\psi_5\rangle + i|\psi_6\rangle) \quad (7)$$

After emerging from the beam splitter BS3, the state is given by

$$|\psi\rangle_{BS3} = \frac{1}{2\sqrt{2}}[(-e^{i\eta}(1 + e^{i\phi}) + (1 - e^{i\phi}))|\psi_5\rangle + i(e^{i\eta}(1 + e^{i\phi}) + (1 - e^{i\phi}))|\psi_6\rangle]|\chi\rangle \quad (8)$$

Let us suppose that the phase η can take two different values, say η_1 and η_2 and likewise the other phase ϕ can

take two different values ϕ_1 and ϕ_2 respectively. The four different forms of the state (8) corresponding to four different values of η and ϕ are given by

$$|\psi_{ij}\rangle_{BS3} = \frac{1}{2\sqrt{2}}[(-e^{i\eta_i}(1 + e^{i\phi_j}) + (1 - e^{i\phi_j}))|\psi_5\rangle + i(e^{i\eta_i}(1 + e^{i\phi_j}) + (1 - e^{i\phi_j}))|\psi_6\rangle]|\chi\rangle \quad (9)$$

where $i, j = 1, 2$.

In order to encode more than four distinguishable outcomes through our scheme using the state (8), one requires at least three pairs of the above choice of states (9) to be orthogonal (Note that the parameter δ characterizing the spin state χ can take one of two values, as earlier). Let us assume that such a configuration is possible for the set $\{|\psi_{11}\rangle, |\psi_{12}\rangle, |\psi_{21}\rangle\}$. Therefore, taking the appropriate inner products of the states (9), one finds that the requirement of $\langle\psi_{11}|\psi_{12}\rangle = 0 = \langle\psi_{11}|\psi_{21}\rangle$ leads to the choice $\phi_2 - \phi_1 = \pi$ and $\eta_2 - \eta_1 = \pi$. However, it follows that $\langle\psi_{12}|\psi_{21}\rangle \neq 0$ in this case, thus violating our assumption of more than two orthogonal states. Further, it is possible to verify using similar arguments that no other combination of more than two states from the set (9) is orthogonal. Therefore, one is unable to encode more than four distinguishable outcomes, or two bits of information, by extending our scheme using more than two beam splitters.

Probabilistic dense coding

Our above analysis pertains to a scheme of deterministic dense coding using the spin and path variables of a single spin-1/2 particle whose state can exist in a superposition of two paths. It could be pertinent to ask here if one could do any better by some scheme which transmits information not exactly, but probabilistically. A variant of this question could be to consider an initial spin state which is not polarized along a certain direction (say) z , but is rather given by

$$|\chi\rangle = \alpha|\rightarrow\rangle_x + \beta e^{i\delta}|\leftarrow\rangle_x \quad (10)$$

and ask, how such a state would fare in dense coding.

The total path-spin state of the particle emerging from the beam splitter BS2 is in this case given by

$$|\psi\rangle_{BS2} = \frac{1}{2}[i|\psi_3\rangle(1 + e^{i\phi}) + |\psi_4\rangle(1 - e^{i\phi})][\alpha|\rightarrow\rangle + \beta e^{i\delta}|\leftarrow\rangle] \quad (11)$$

Corresponding to different values of the parameter ϕ and δ choosing from the set $\{0, \pi\}$, the output states which are sent to Bob are given by (i) $|\psi\rangle_{BS2} = i|\psi_3\rangle \otimes (\alpha|\rightarrow\rangle + \beta|\leftarrow\rangle)$, if $\phi = 0, \delta = 0$; (ii) $|\psi\rangle_{BS2} = |\psi_4\rangle \otimes (\alpha|\rightarrow\rangle + \beta|\leftarrow\rangle)$, if $\phi = \pi, \delta = 0$; (iii) $|\psi\rangle_{BS2} = i|\psi_3\rangle \otimes (\alpha|\rightarrow\rangle - \beta|\leftarrow\rangle)$, if $\phi = 0, \delta = \pi$; and (iv) $|\psi\rangle_{BS2} = |\psi_4\rangle \otimes (\alpha|\rightarrow\rangle - \beta|\leftarrow\rangle)$, if $\phi = \pi, \delta = \pi$.

It is to be noted that the states $|\phi_1\rangle = \alpha|\rightarrow\rangle + \beta|\leftarrow\rangle$ and $|\phi_2\rangle = \alpha|\rightarrow\rangle - \beta|\leftarrow\rangle$ are non-orthogonal states and hence cannot be distinguished deterministically. However, protocols for probabilistic dense coding [17] rely on

the fact that non-orthogonal states can be distinguished with some probability of success if they are linearly independent. Therefore to distinguish the states $|\phi_1\rangle$ and $|\phi_2\rangle$ probabilistically, these should be linearly independent. This follows from the fact that if $\lambda_1|\phi_1\rangle + \lambda_2|\phi_2\rangle = 0$ for some scalars λ_1 and λ_2 , then $\lambda_1 = \lambda_2 = 0$. Thus the non-orthogonal states $|\phi_1\rangle$ and $|\phi_2\rangle$ are linearly independent and hence they can be distinguished with some probability of success.

Now the remaining task is to distinguish the states $|\phi_1\rangle$ and $|\phi_2\rangle$ and to achieve this goal, Bob performs a generalised measurement described by positive operator valued measurements (POVM)s. The corresponding POVM elements for the spin state are given by

$$\begin{aligned} S_1 &= \beta^2 |\rightarrow\rangle\langle\rightarrow| + \alpha\beta(|\rightarrow\rangle\langle\leftarrow| + |\leftarrow\rangle\langle\rightarrow|) + \alpha^2 |\leftarrow\rangle\langle\leftarrow| \\ S_2 &= \beta^2 |\rightarrow\rangle\langle\rightarrow| - \alpha\beta(|\rightarrow\rangle\langle\leftarrow| + |\leftarrow\rangle\langle\rightarrow|) + \alpha^2 |\leftarrow\rangle\langle\leftarrow| \\ S_3 &= (1 - 2\beta^2)|\rightarrow\rangle\langle\rightarrow| + (1 - 2\alpha^2)|\leftarrow\rangle\langle\leftarrow| \end{aligned} \quad (12)$$

where $S_1 + S_2 + S_3 = I$.

For illustration of this scheme, let us suppose that Bob is given the state $|\psi\rangle_{BS2} = i|\psi_3\rangle \otimes (\alpha|\rightarrow\rangle + \beta|\leftarrow\rangle) \equiv i|\psi_3\rangle \otimes |\phi_1\rangle$. He then performs the measurement on the spin state $|\phi_1\rangle$ described by $\{S_1, S_2, S_3\}$. The POVM element S_2 is chosen in such a way that $\langle\phi_1|S_2|\phi_1\rangle = 0$ and this indicates the fact that the probability of getting the result S_2 is zero when the state $|\phi_1\rangle$ is given. Hence, the measurement outcome may be either S_1 or S_3 . If he gets S_1 then the state is surely $|\phi_1\rangle$, but if he gets S_3 , the result is inconclusive. The success probability of distinguishing $|\phi_1\rangle$ is $1 - \langle\phi_1|S_3|\phi_1\rangle = 2(1 - 2\alpha^2\beta^2)$. If $\alpha = \beta = \frac{1}{\sqrt{2}}$, this reduces to the case of deterministic dense coding described earlier, with the success probability being unity.

It may be noted here that introducing additional beam splitters does not help in encoding more information even probabilistically, since there do not exist more than two linearly independent vectors in the two-dimensional Hilbert space that is relevant for our spin-1/2 particle. Nevertheless, one could split further the two channels $|\psi_3\rangle$ (into say, $|\psi_5\rangle$ and $|\psi_6\rangle$), and $|\psi_4\rangle$ (into say, $|\psi_7\rangle$ and $|\psi_8\rangle$) by introducing beam-splitters in both of them, and then use additional Stern-Gerlach measurement devices at the various channels. However, such a scheme would correspond to creating more than two orthogonal path states (or effectively qubits) per particle, and additional information may thereby be encoded. In the present analysis we have restricted ourselves to consider two effective qubits (corresponding to a two-level spin state and a two-level path state) for a single particle.

Discussion

Note that, in the usual dense coding scheme [10, 11, 13], the path degrees of freedom of an individual spin-1/2 system are implicitly utilized in physically transporting one member of the EPR pair (that has the communicated information encoded in its spin state) to a receiver pos-

sessing the other member of the entangled pair, who then performs Bell-basis measurement to discern the transferred information. In the standard dense coding protocols using entangled states, though the path variables of a particle are implicitly involved in the very act of physically sending the particle from one location to another, these path variables are never explicitly utilized in the mechanism of encoding information. On the other hand, in the scheme we propose here, while the EPR-Bohm entangled state is not required, it is in the act of physically sending a spin-1/2 particle that we use a variant of the Mach-Zehnder interferometer involving suitable manipulations of both the path and spin degrees of freedom.

In our scheme we exploit the ubiquitously present position coordinates of the particle used for dense coding. Thus, we are able to transmit information about four distinguishable outcomes using both the spin and the suitably superposed path degrees of freedom of a single particle. The single spin-1/2 particle that we use, in effect, carries two qubits with it (one through its spin, and another through the two possible paths emergent from the beam splitter). Therefore, our scheme does not violate the Holevo bound, albeit performing the task of dense coding without quantum entanglement. The analysis of the probabilistic variant of our dense coding scheme further reinforces the notion that the two effective qubits carried by the position and path variables of the spin-1/2 particle can encode, at best, two bits of information. We conclude by emphasizing that our discussion of the proposed scheme, though presented in terms of neutral spin-1/2 particles (such as neutrons), works equally well for photons with appropriate polarising and analysing devices.

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